COMPARISON OF TWO LINEAR MODELS OF SOIL MOISTURE DYNAMICS

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The results of a comparison of two models of the motion of soil moisture are presented. It is shown that under certain conditions one of them describes the motion of moisture in the direction of a moisture gradient.

The dynamics of soil moisture have traditionally been described by an equation of the diffusion-equation type, which corresponds to the so-called diffusion model. Certain soil column experiments, however, cannot be explained by the diffusion model. Recent study of the motion of soil moisture has led to the creation of a new model based on representation of the soil as a cracked-porous medium. The new model is a generalization of the diffusion model and is consistent with the above-mentioned experiments.

The differential equation for the moisture content, obtained on the basis of this model, is not new and has already been studied in connection with the motion of a fluid in cracked media [1]. However, the motion of moisture in the direction of moisture gradient, detected experimentally in soil columns, has received little attention. The essence of this effect was described, for example, in [2], where, incidentally, qualitative correspondence between the new model and the motion of the moisture along the gradient was noted.

The effect in question consists briefly in the following. If the initial distribution of moisture content in the column is nonuniform with respect to the coordinate and intense evaporation takes place at the wet end, the moisture content of the relatively dry end decreases during the initial drying period, despite the fact that the moisture gradient is still directed toward the wet end. It should be noted that this decrease in moisture content at the dry end is important from the standpoint of a model comparison only at small times, since when the moisture gradient changes sign with time the decrease in moisture content at that end of the specimen becomes a simple consequence of evaporation.

We consider the following boundary value problem, giving the moisture content u(t, x) as a function of time t and coordinate x on the finite interval $0 \leq x \leq H$:

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + a^{-2}\frac{\partial^3 u}{\partial t\,\partial x^2}$$

$$u(0, x) = \varphi(x), \qquad Q(t, x) \equiv D\frac{\partial u}{\partial x} + a^{-2}\frac{\partial^2 u}{\partial t \partial x} \tag{1}$$

$$\delta Q(t, 0) - \alpha u(t, 0) = f_1, \qquad \gamma Q(t, H) + \beta u(t, H) = f_2 .$$

Obviously, problem (1) is a generalization of the corresponding problem formulated for the diffusion equation and goes over into the latter when $a^{-2} = 0$ $(a = \infty)$. All the constants D, a, α, β, γ, δ, f_1, f_2 are assumed nonnegative. As noted above, only small values of t are meaningful. Therefore it is natural to assume that the quantities α, β, γ, δ, f_1, f_2 are constant. We also note that $Q(t, 0) > 0$ corresponds to evaporation from the surface $x = 0$ and $Q(t, H) > 0$ corresponds to moistening at the level $x = H$.

Problem (1) can be made solvable by using the classical Fourier method, as used in [2] in connection with a less general case. The uniqueness of the solution of linear problem (1) with variable coefficients is demonstrated by means of energy inequalities.

To be specific, let us consider the variation of the moisture content with time u(t, x) at the level $x = H$. It is easy to see, first of all, that if $\gamma = 0$, then $u(t, H) = f_2/\beta$, and the problem is a trivial one. Accordingly, it may be assumed that $\gamma > 0$ and, consequently, without loss of generality, $\gamma = 1$. Problem (1) is solved by means of a Laplace-Carson transformation with respect to the variable t. In this case it is sufficient to compute the operational transform v(p) of the function u(t, H). Then, finding the limits

$$\lim v(p) = \varphi(H), \quad \lim p\,[v(p) - \varphi(H)] = \lambda \quad \text{as} \quad p \to \infty$$

we obtain the first two terms of the expansion of u(t, H) in powers of t for small t:

$$u(t, H) = \varphi(H) + \lambda t + o(t), \quad \lambda = \frac{\partial}{\partial t} u(t, H) \quad \text{at} \quad t = 0. \tag{2}$$

It should also be noted that when $a \neq \infty$ the expansion of v(p) in negative powers of p lacks a term containing $p^{-0.5}$, so that in the expansion of u(t, H) there is no term $c\sqrt{t}$. The same applies to any powers of t less than unity. Thus, in expression (2) the number λ is of principal interest from the standpoint of the modeled effect. Omitting the rather clumsy calculations, we can write λ in the case $\delta = 1$ (which is obviously equivalent to $\delta > 0$) in the form:

$$\lambda = \frac{a}{\operatorname{sh} aH}\left[(f_2 - \beta\varphi(H))\operatorname{ch} aH - (f_1 + \alpha\varphi(0)) - aD\int_0^H \varphi'(y)\operatorname{sh} ay\, dy\right] \tag{3}$$

or ($0 < a < \infty$)

$$\lambda = \frac{a}{\operatorname{sh} aH}\left[Q(0, H)\operatorname{ch} aH - Q(0,0) - aD\int_0^H \varphi'(y)\operatorname{sh} ay\, dy\right]. \tag{4}$$

We first consider the case in which

$$f_1 + \alpha\varphi(0) - D\varphi'(0) = f_2 - \beta\varphi(H) - D\varphi'(H) = 0. \tag{5}$$

This corresponds to the requirement that for $0 < a < \infty$ the boundary conditions of problem (1) at t = 0 have the form

$$\left.\frac{\partial^2 u}{\partial t\, \partial x}\right|_{x=0} = \left.\frac{\partial^2 u}{\partial t \partial x}\right|_{x=H} = 0.$$

If, however, we consider the diffusion model ($a = \infty$), then Eqs. (5) imply the continuity of the boundary conditions of problem (1) at t = 0.

To determine the rate of growth of u(t, H) at t = 0 in the case of the diffusion model (we denote it by λ_0), we can pass to the limit as $a \to \infty$ in Eq. (3), having first integrated by parts. We have

$$\lambda_0 = D\varphi''(H). \tag{6}$$

Equation (6) shows that the rate of variation of the moisture content at the end x = H is initially determined by the diffusion coefficient D and the properties of the initial distribution $\varphi(x)$ at only one point x = H, if the model is diffusional. On the other hand, as shown by Eq. (4), if the model is not diffusional, the rate depends on all the values of $\varphi(x)$ (H is a functional of $\varphi'(x)$), the initial flows Q(0, 0) and Q(0, H), the length H, and, of course, the parameters of the equation D and a. Briefly, Eq. (6) expresses the dependence on the local properties of the parameters of the problem (incidentally only on the second derivative of the initial distribution at a single point), whereas (4) expresses the dependence on the integral properties. In this way it is possible to determine the basic difference between the models considered.

The fact that λ_0 is independent of the flow Q(0, H) prompts the following comment. In the case in question ($a = \infty$)

$$Q(0, H) = \frac{\partial}{\partial x} u(0, H), \qquad \lambda_0 = \frac{\partial}{\partial t} u(0, H).$$

Thus, Q(0, H) characterizes the shape of the u(0, x) curve at the point x = H, while λ_0 characterizes the variation of moisture content with time, i.e., the shape of the u(t, H) curve at the point t = 0.

Integrating by parts in Eq. (3) and using (5), we can obtain a more compact expression for λ:

$$\lambda = \frac{aD}{\operatorname{sh} aH}\int_0^H \varphi''(y)\operatorname{ch} ay\, dy.$$

Comparing the latter equation with (6), we see that it differentiates the models even more sharply: whereas λ_0 depends on the value of φ'' at a single point, λ depends on all the values of $\varphi''(x)$. This becomes especially apparent if it is assumed that the parameter a is sufficiently small (i.e., the coefficient a^{-2} is large). Then

$$\lambda \approx \frac{1}{H}\int_0^H D\varphi''(y)\, dy;$$

here λ is the mean of the function $D\varphi''(x)$ on the interval [0, H], whereas λ_0 is the value of the same function at the point x = H.

We consider a few particular cases.

At x = 0 let the evaporation Q(t, 0) = $f_1 > 0$, while at the end x = H let Q(t, H) = 0. This means that in problem (1) $\alpha = \beta = f_2 = 0$ is given. It then follows from (5) that $\varphi'(H) = 0$ and, assuming $\varphi'(x) \leq 0$, we obtain $\varphi''(x) > 0$. Thus, in this case for the diffusion model we have the inequality $\lambda_0 > 0$, which indicates an increase in moisture content at x = H at the initial instant. On the other hand, from (3) we obtain

$$\lambda = \frac{a}{\operatorname{sh} aH}\left[aD\int_0^H (-\varphi'(y)) \operatorname{sh} ay\, dy - f_1\right]. \tag{7}$$

Equation (7) shows that λ becomes negative (if $a > 0$ is sufficiently small), i.e., that at the relatively dry end (x = H) the moisture may decrease, despite the fact that the moisture gradient is directed toward the point x = 0. It is precisely this effect that distinguishes model (1) from the diffusion model and that has been detected experimentally (we have in mind the given particular case of the boundary value problem).

Since $\lim \lambda = \lambda_0 > 0$ as $a \to \infty$, it is possible to find a value of the coefficient a for which $\lambda = 0$, i.e., a threshold value a_1 such that at $a \geq a_1$ the effect is not simulated. For this it is sufficient to solve for a in the equation

$$aD\int_0^H \varphi'(y) \operatorname{sh} ay\, dy = -f_1.$$

If a_1 is sufficiently small, it can be calculated approximately from the relation

$$a_1^{-2} \approx -\frac{D}{f_1}\int_0^H y\varphi'(y)\, dy = \frac{D}{f_1}\int_0^H [\varphi(y) - \varphi(H)]\, dy. \tag{8}$$

The physical significance of relation (8) is as follows. The greater the diffusion coefficient D and the total excess of the moisture content of the specimen as a whole over the moisture content of its relatively dry end and the less intense the evaporation, the higher the values of a^{-2} at which the effect of motion of moisture along the moisture gradient is observed.

It follows from (4) that in a medium in which the influence of diffusivity can be neglected, the effect in question is determined only by the values of the initial flows at the points x = 0 and x = H. In fact, in this case it is possible to set D = 0 in (4), whence follows the assertion.

If, however, the diffusion mechanism predominates, the parameter a may be assumed fairly large. By means of simple calculations, from (3) we then obtain

$$\lambda = D\varphi''(H) - 0.5a^{-1}D\varphi'''(H) + O(a^{-2}). \tag{9}$$

Naturally, the principal part of this representation is λ_0, and therefore in this case motion along the moisture gradient is not to be anticipated. We note that at large a the "finer" effect is not simulated either. To be specific, again let $\alpha = \beta = f_2 = \varphi'(H) = 0$ and $\varphi'(x) < 0$. Then $\lambda_0 > 0$ and the fact that $\lambda < \lambda_0$ at $1 \ll a < \infty$ may be called the "finer" effect. However, as follows from (9), this depends on the sign of $\varphi'''(H)$, which may be arbitrary, and does not affect the sign of λ in (7). Thus, problem (1) simulates the effect only at values of a lying on a certain finite interval $0 < a_1 \leq a \leq a_2 < \infty$.

An expansion analogous to (2) can also be written for u(t, 0). For this it is necessary to substitute H–x for x in problem (1) and use (3):

$$u(t, 0) = \varphi(0) + \lambda_1 t + o(t)$$
$$\lambda_1 = \frac{a}{\operatorname{sh} aH}\left[Q(0, H) - Q(0,0) \operatorname{ch} aH + aD\int_0^H \varphi'(y) \operatorname{sh} a(H-y)\, dy\right]. \tag{10}$$

By means of Eqs. (10) it is possible to obtain a particular case of Eq. (4) on the assumption that $\delta = 0$. In fact, if $\delta = 0$, then from the first boundary condition in (1) it follows that $u(t, 0) \equiv \varphi(0)$ = const; therefore $\lambda_1 = 0$. Finding from the equation $\lambda_1 = 0$ the value of Q(0, 0) and substituting it into (4), we obtain for $\delta = 0$ the value of λ (figuring in (2))

$$\lambda = \frac{a}{\operatorname{ch} aH}\left[Q(0, H) \operatorname{sh} aH - aD\int_0^H \varphi'(y) \operatorname{ch} ay\, dy\right]. \tag{11}$$

Equation (11) shows that the motion of liquid along the gradient at the point $x = H$ depends essentially on the presence of evaporation at the other end of the specimen $x = 0$. In fact, if one wishes to detect the effect, it is necessary to set $Q(0, H) \geq 0$, since otherwise $Q(0, H) < 0$ and a decrease in the moisture content $u(t, H)$ may be simply a consequence of the removal of moisture. But then, when $\varphi'(x) < 0$, irrespective of the value of the parameter a, we have the inequality $\lambda > 0$, $(\delta = 0)$. Admittedly, even in this case, it is possible to obtain a negative right-hand side of (11) as a result of a special choice of $\varphi(x)$, while retaining, however, the inequality $\varphi(H) < \varphi(0)$. Such a function $\varphi(x)$ should have first a minimum and then a maximum as x approaches the point H, a condition that is probably rather difficult to satisfy experimentally. Moreover, at large moisture gradients in the specimen the assumption of a constant diffusion coefficient D is no longer physically justifiable. We note that all known experimental verifications of the effect in question have been obtained in the presence of intense evaporation from the surface $x = 0$.

Finally, let condition (5) not be satisfied. Then Eqs. (3) and (4) are preserved at $a < \infty$. However, at $a = \infty$ (diffusion model) Eq. (2) takes the form:

$$u(t, H) = \varphi(H) + \lambda_0 t + [f_2 - \beta\varphi(H) - D\varphi'(H)]\left(\frac{2\sqrt{t}}{\sqrt{\pi D}} - \frac{\alpha t}{D^2}\right) + o(t).$$

Hence it follows that the increase or decrease in $u(t, H)$ at small t, though it still depends on the parameters of the initial distribution only at the single point $x = H$, proceeds more rapidly than before, since $\sqrt{t} \gg t$.

In conclusion, we note that, knowing the parameters of problem (1) and measuring λ, we can determine the coefficient a^{-2} in Eq. (1) from (3) (or (4)).

REFERENCES

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